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STRUCTURES AND MATERIALS RESEARCH

DEPARTMENT OF CIVIL ENGINEERING

FINITE ELEMENT SOLUTION OF AXISYMMETRICAL DYNAMIC PROBLEMS OF SHELLS OF REVOLUTION

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Preface

The investigation reported herein is a part of the research carried out under the sponsorship of National Aeronautics and Space Administration. Research Grant No. NsG-274, Supplement No. 2.

The research described in this report was conducted under the supervision and technical responsibility of Egor P. Popov, Professor of Civil Engineering, Division of Structural Engineering and Structural Mechanics, University of California, Berkeley, California. Mr. H.Y. Chow, Graduate Student, Division of Structural Engineering and Structural Mechanics, was responsibile for carrying out the detailed derivations.

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Mrs. M. French typed the final report, and Mr. B. Kot prepared the drawings.

Synopsis

A finite element solution for the natural frequencies and mode shapes of free axisymmetrical vibrations and the dynamic response of arbitrary rotationally symmetric shells is presented in this report. The stiffness for the basic annular plate, conical and cylindrical elements are constructed by using the static homogeneous solutions of the classical plate and shell bending theories. The mass matrix for the annular plate is based on the same solutions. For the conical and the cylindrical elements the mass matrix is determined using an assumed displacement field. The developed solutions were programmed in Fortran IV. The illustrative examples include a detailed analysis of a problem of a shallow spherical cap subjected to an axisymmetrical pressure varying with time, a solution for a plate subjected to a time-dependent ring load and an analysis of a dynamic response of a complete sphere.

I. INTRODUCTION

Thin shells of revolution are the most important structures used in the aero-space industry. Much work has been done on the solution of static problems for such shells, however, the dynamic analysis of shells of revolution has been principally limited to the consideration of special cases. The treatment of the general dynamic analysis of thin elastic shells according to the bending theory has not been resolved satisfactory. Federhofer [1] in 1937 and E. Reissner [2] in 1946 were the first people to treat the axisymmetric vibration problem of shallow spherical shells by means of a variational method. Naghdi and Kalmins [3], using an uncoupled system of equations equivalent to that in [1], obtained in exact solution for the problem of axisymmetric natural frequencies and mode shapes of free-vibration of a hemispherical shell. Subsequently, a general numerical analysis of the natural frequencies and mode shape of free-vibration of rotational shells was given by Kalkins [4]. Recently Klein [5], using the matrix displacement finite element approach, has succeeded in treating the dynamic response problem for an arbitrary shell of revolution. In [5], for the basic conical element the mass and stiffness matrix used in the equations of motion were derived from an assumed displacement field taken in polynomial form. The system of equation was solved by means of a finite difference technique to give directly the dynamic response.

The purpose of the present report is to determine the axisymmetrical natural frequencies and mode shapes of free-vibration as well as of the dynamic response to arbitrary axisymmetrical loading for rotationally symmetric shells with various boundary conditions. The analysis is based on the linear bending theory of shells. Any shell studied by the proposed procedure is approximated by the combination of basic shell elements of trucated conical, cylindrical and plate segments. The stiffness matrix is derived from the analytical solution of the homogeneous field equations of the static shell theory for the basic elements. These solutions are also used as the displacement functions to formulate the element mass matrix in a manner similar to the Rayleigh-Ritz procedure. The equations of motions are solved by using normal-mode superposition approach. To carry out those solutions, computer programs have been wirtten in Fortran IV.

As an illustration of the developed solution, the problem of a spherical cap subjected to a axisymmetrical pressure load varying with time is solved. The results for dynamic response are compared with those which Klein [5] obtained by a different method. Excellent agreement between the two solutions is found. In order to show the characteristic dynamic behavior of the shell the natural frequencies and mode shapes are also given. Furthermore, a circular plate under time-dependent ring load is analyzed by using only two finite elements to indicate the advantage of "exact" formulations in the proposed solution. The results are identical with those using 20 finite elements. Finally, to demonstrate the generality of the proposed method the dynamic response of a complete sphere is solved by using 50 elements.

II. HOMOGENEOUS STATIC SOLUTIONS FOR BASIC SHELL ELEMENTS

The static homogeneous solutions for axisymmetrical deformation of uniform thickness basic shell elements are available in Refs. 6, 7, 10 and 11. In this report we use these solutions as a basis for formulating the element stiffness matrices [k] and the element mass matrices [m]. It is advantageous to express these solutions in matrix form as follows:

$$\left\{ d_i(s) \right\} = \begin{Bmatrix} \chi(s) \\ v(s) \\ w(s) \end{Bmatrix} = [X_{ij}(s)] \left\{ A_j \right\} \tag{1}$$

and

$$\left\{ S_i(s) \right\} = \begin{Bmatrix} M_s(s) \\ N_s(s) \\ Q_s(s) \end{Bmatrix} = [Y_{ij}(s)] \left\{ A_j \right\} \tag{2}$$

where $i = 1,2,3$ and $j = 1,2,......,6$.

In these equations $\left\{d(s)\right\}$ are displacement-variables which are comprised of rotational $\chi(s)$, meridional $v(s)$, and normal $w(s)$ displacements; $\left\{S(s)\right\}$ are force-variables which consist of meridional moments $M_s(s)$, meridional stress-resultants $N_s(s)$, and shearing stress-resultants $Q_s(s)$. $[X(s)]$ and $[Y(s)]$ are 3 by 6 matrices whose rows represent six linearly independent coefficients of static solutions. $\left\{A\right\}$ is a constant column matrix which can be determined from the nodal displacements at each end of the shell element.

The non-zero elements functions X_{ij} and Y_{ij} of $[X(s)]$ and $[Y(s)]$, for each of the basic shell elements used in this work, are listed separately in the following sections.

II-1. Conical Elements

The non-zero functions X_{ij} and Y_{ij} of $[X(s)]$ and $[Y(s)]$ in Eqs. (1) and (2) for the conical element are listed as below. The symbols involved are defined in Fig. 1.

$X_{11}=c_1(\text{bei } y + 2y^{-1}\text{ber}' \, y)$; $X_{12}=-c_1(\text{ber } y - 2y^{-1}\text{bei}' \, y)$;

$X_{13}=c_1(\text{kei } y + 2y^{-1}\text{ker}' \, y)$; $X_{14}=-c_1(\text{ker } y - 2y^{-1}\text{ker}' \, y)$;

$X_{15}=-(\cot \alpha/Et)\ (1/s)$; $X_{21}=c_2[(\nu/2)\text{ber } y - (1+\nu)y^{-1}\text{bei}' \, y]$;

$X_{22}=c_2(\frac{\nu}{2}\ \text{bei } y + (1+\nu)\ y^{-1}\ \text{ber}' \, y)$; $X_{23}=[\frac{\nu}{2}\ \text{ker } y-(1+\nu)y^{-1}\text{kei}' \, y]$;

$X_{24}=c_2(\frac{\nu}{2}\ \text{kei } y+(1+\nu)y^{-1}\ \text{ker}' \, y)$; $X_{25}= \log\ (s)\ /Et$;

$X_{26}= 1$; $X_{31}=c_3(\text{ber } y-\frac{1}{2}\ y\ \text{bei}' \, y)$;

$X_{32}=c_3(\text{bei } y-\frac{1}{2}\ y\ \text{bei } y)$; $X_{33}=c_3(\text{ker } y-\frac{1}{2}\ y\ \text{ker}' \, y)$;

$X_{34}=c_3(\text{kei } y-\frac{1}{2}\ y\ \text{kei}' \, y)$; $X_{35}=-\ \frac{\cot\alpha}{Et}\ (\log\ s\ +\ \nu)$; $X_{36}=-\cot\ \alpha$;

$Y_{11}=c_4(y\text{bei}'y-2(1-\nu)\ (\text{bei } y+2y^{-1}\text{ber}'y)\ )$;

$Y_{12}=-c_4(y\text{ber}'y-2(1-\nu)\ (\text{ber } y-2y^{-1}\text{bei}'y)\ )$;

$Y_{13}=c_4(y\text{kei}'y-2(1-\nu)\ (\text{kei } y+2y^{-1}\text{ker}'y)\ )$;

$Y_{14}=-c_4(y\text{ker}'y-2(1-\nu)\ (\text{ker } y-2y^{-1}\ \text{ker}'y)\ )$;

$Y_{21}=c_5(\text{ber } y-2y^{-1}\text{bei}'y)$; $Y_{22}=c_5(\text{bei } y+2y^{-1}\text{ ber}'y)$;

$Y_{23}=c_5(\text{ker } y - 2y^{-1}\text{ kei}'y)$; $Y_{24}=c_5(\text{kei } y + 2y^{-1}\text{ ker}'y)$;

$Y_{25}=\frac{1}{s}$; $Y_{31}=+c_6(\text{ber } y - 2y^{-1}\text{ bei}'y)$;

$Y_{32}=c_6(\text{bei } y + 2y^{-1}\text{ ber}'y)$; $Y_{33}=c_6(\text{ker } y - 2y^{-1}\text{ kei}'y)$;

$Y_{34}=c_6(\text{kei } y + 2y^{-1}\text{ ker}'y)$;

Where $c_1 = \dfrac{2\sqrt{3(1-\nu^2)}}{E\ h^2}\cot\alpha$ $\quad c_2 = +\dfrac{2\cot\alpha}{Eh}$

$c_3 = +\dfrac{\cot^2\alpha}{Eh}$ $\quad c_4 = 2\,y^{-2}$

$c_5 = -\dfrac{1}{s}\cot\alpha$ $\quad c_6 = \dfrac{1}{s}$

II-2. Cylindrical Elements

The non-zero functions X_{ij} and Y_{ij} of $[X(s)]$ and $[Y(s)]$ in Eqs. (1) and (2) for the cylindrical elements are listed below and the symbols involved are defined in Fig. 2.

$X_{11}=c_7(\cos\kappa\xi + \sin\kappa\xi)$ $\quad X_{12}=-c_7(\cos\kappa\xi - \sin\kappa\xi)$

$X_{13}=c_8(\cos\kappa\xi - \sin\kappa\xi)$ $\quad X_{14}=c_8(\cos\kappa\xi + \sin\kappa\xi)$

$X_{21}=c_9(\sin\kappa\xi - \cos\kappa\xi)$ $\quad X_{22}=-c_9(\sin\kappa\xi + \cos\kappa\xi)$

$X_{23}=c_{10}(\sin\kappa\xi + \cos\kappa\xi)$ $\quad X_{24}=c_{10}(\sin\kappa\xi - \cos\kappa\xi)$

$X_{25} = \dfrac{s}{Eh}$ $\quad X_{26} = 1$

$X_{31}=e^{-\kappa\xi}\cos\kappa\xi$ $\qquad$ $X_{32}=e^{-\kappa\xi}\sin\kappa\xi$

$X_{33}=e^{\kappa\xi}\cos\kappa\xi$ $\qquad$ $X_{34}=e^{\kappa\xi}\sin\kappa\xi$

$X_{35}=-\dfrac{\nu d}{Eh}$

$Y_{11}=c_{11}\sin\kappa\xi$ $\qquad$ $Y_{12}=-c_{11}\cos\kappa\xi$

$Y_{13}=c_{12}\sin\kappa\xi$ $\qquad$ $Y_{14}=-c_{12}\cos\kappa\xi$

$Y_{25}=1$ $\qquad$ $Y_{31}=c_{13}(\cos\kappa\xi-\sin\kappa\xi)$

$Y_{32}=c_{13}(\cos\kappa\xi+\sin\kappa\xi)$ $\qquad$ $Y_{33}=c_{14}(\cos\kappa\xi+\sin\kappa\xi)$

$Y_{34}=-c_{14}(\cos\kappa\xi-\sin\kappa\xi)$

Where

$$c_7=-\frac{\kappa}{d}e^{-\kappa\xi} \qquad c_8=\frac{\kappa}{d}e^{\kappa\xi}$$

$$c_9=-\frac{\nu}{2\kappa}e^{-\kappa\xi} \qquad c_{10}=-\frac{\nu}{2\kappa}e^{-\kappa\xi}$$

$$c_{11}=\frac{2D\kappa^2}{d^2}e^{-\kappa\xi} \qquad c_{12}=-\frac{2D\kappa^2}{d^2}e^{\kappa\xi}$$

$$c_{13}=-\frac{2D\kappa^3}{d^3}e^{-\kappa\xi} \qquad c_{14}=\frac{2D\kappa^3}{d^3}e^{\kappa\xi}$$

II-3. Plate Elements

The non-zero functions X_{ij} and Y_{ij} of [X(s)] and [Y(s)] in Eqs. (1) and (2) for the plates elements are listed as below and the symbols involved are defined in Fig. 3.

$X_{11}=-s(1+2\ \log\ s)$ $\quad X_{12}=-2s$ $\quad X_{13}=-\frac{1}{s}$

$X_{25}=\frac{1}{s}$ $\quad X_{26}=s$ $\quad X_{31}=s^2\ \log\ s$

$X_{32}=s^2$ $\quad X_{33}=\log\ s$ $\quad X_{34}=1$

$Y_{11}=D[(3+\nu)+2(1+\nu)\ \log\ s]$ $\quad Y_{12}=2D\ (1+\nu)$

$Y_{13}=-D(1-\nu)\ \frac{1}{s^2}$ $\quad Y_{25}=-\frac{Eh}{(1+\nu)}\ \frac{1}{s^2}$

$Y_{26}=\frac{Eh}{1-\nu}$ $\quad Y_{31}=-\frac{4D}{s}$

III. BASIC SHELL ELEMENT STIFFNESS MATRIX

In this report the derivation of the stiffness matrix [k] for each of the basic shell elements is based upon the static homogeneous solutions given in Eqs. (1) and (2). In these equations $\left\{ A \right\}$ must be determined from a total of six boundary conditions, i.e., from three conditions at each end of the shell elements. This is done by evaluating Eq. (1) at s=a, and s=b. Thus:

$$\left\{ A \right\} = \left[\frac{X\ (a)}{X\ (b)} \right]^{-1} \left\{ \frac{d\ (a)}{d\ (b)} \right\} \tag{3}$$

where we define the elements of the matrix $\left\{ \frac{d\ (a)}{d\ (b)} \right\}$ in Eq. (3) as the shell element local displacement coordinates $\left\{ d \right\}$, (see Fig. 4). On this basis we can express the displacement and force-variables in terms of six local displacements $\left\{ d \right\}$ by substituting $\left\{ A \right\}$ from Eq. (3) into Eq. (1), and Eq. (2). This yields

$$\left\{ d\ (s) \right\} = [X(s)] \left[\frac{X\ (a)}{X\ (b)} \right]^{-1} \left\{ d \right\} \tag{4}$$

and

$$\left\{ S(s) \right\} = \left[Y(s) \right] \left[\frac{X\ (a)}{X\ (b)} \right]^{-1} \left\{ d \right\} \tag{5}$$

Evaluating $\left\{ S(s) \right\}$ in Eq. (5) at the edges of the element s=a, and s=b, we obtain

$$\left\{ \frac{S\ (a)}{S\ (b)} \right\} = \left[\frac{Y\ (a)}{Y\ (b)} \right] \left[\frac{X\ (a)}{X\ (b)} \right]^{-1} \left\{ d \right\} \tag{5a}$$

To determine the element stiffness matrix, we have to find a relation between the six local displacements $\left\{d\right\}$ and the corresponding forces $\left\{S\right\}$ in the local coordinate system (see Fig. 4). By comparing Fig. 1 and Fig. 4, we find the relation between $\left\{S\right\}$ and $\left\{\begin{array}{c} S\ (a) \\ \hline S\ (b) \end{array}\right\}$:

$$\left\{S\right\} = \left[\begin{array}{c|c} -I & 0 \\ \hline 0 & I \end{array}\right] \left\{\begin{array}{c} S\ (a) \\ \hline S\ (b) \end{array}\right\} \tag{6}$$

Where [I] is identity matrix and [0] is null matrix. By substiuting Eq. (5a) into the above equation, we can find the desired relation between the shell element local coordinates $\left\{d\right\}$ and the forces $\left\{S\right\}$ in the same coordinates. That is:

$$\left\{S\right\} = \left[\begin{array}{c|c} -I & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c} Y\ (a) \\ \hline Y\ (b) \end{array}\right] \left[\begin{array}{c} X\ (a) \\ \hline X\ (b) \end{array}\right]^{-1} \left\{d\right\} \tag{7}$$

From definition it follows that the matrix $\left[\begin{array}{c|c} -I & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c} Y\ (a) \\ \hline Y\ (b) \end{array}\right] \left[\begin{array}{c} X\ (a) \\ \hline X\ (b) \end{array}\right]^{-1}$ in Eq. (7) is the shell element stiffness matrix based on local coordinate system $\left\{d\right\}$. Since as a final result we want to form a single structural stiffness matrix in cooperating many elements, we transform the local element stiffness matrix to a stiffness matrix based on the system (or global) coordinates $\left\{q\right\}$ (see Fig. 5). The transformation relation between local coordinates $\left\{d\right\}$ and the system coordinates $\left\{q\right\}$ can be obtained by comparing Fig. 4 and Fig. 5. This relation can be expressed by the following equation

$$\left\{d_i\right\} = [T_{ij}] \left\{q_j\right\} \tag{8}$$

where i, j=1,2,......, 6 and:

$$[T] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & \cos\alpha & -\sin\alpha & & 0 & \\ 0 & \sin\alpha & \cos\alpha & & & \\ \hline & & & 1 & 0 & 0 \\ & 0 & & 0 & \cos\alpha & -\sin\alpha \\ & & & 0 & \sin\alpha & \cos\alpha \end{array}\right] \tag{8a}$$

Furthermore, the relation between forces $\{S\}$ in the $\{d\}$ coordinate system, and forces $\{Q\}$ in the $\{q\}$ coordinate system can be expressed as:

$$\{Q_i\} = [T_{ji}]^T \{S_j\} \tag{9}$$

Substituting Eq. (7) and Eq. (8) into the above equation, we have

$$\{Q\} = [T]^T \left[\begin{array}{c|c} -I & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c} Y\ (a) \\ \hline Y\ (b) \end{array}\right] \left[\begin{array}{c} X\ (a) \\ \hline X\ (b) \end{array}\right]^{-1} [T]\ \{q\} \tag{10}$$

where it is convenient to define the following matrices as:

$$[B] = \left[\begin{array}{c} X\ (a) \\ \hline X\ (b) \end{array}\right] \tag{10a}$$

and:

$$[C] = \left[\begin{array}{c|c} -I & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c} Y\ (a) \\ \hline Y\ (b) \end{array}\right] \tag{10b}$$

The desired element stiffness matrix $[k]$ based on generalized system coordinate $\{q\}$, on the above basis becomes:

$$[k] = [T]^T [C]\ [B]^{-1}\ [T] \tag{11}$$

Since in this work [k] is derived from classical static solutions for thin plates and shells and is not found by assuming a particular displacement field, the stiffness matrix given by Eq. (11) is "exact" for the selected basic shell elements.

IV. BASIC SHELL ELEMENT MASS MATRIX

In dynamic problems the forces and displacements are time dependent, and the inertia of accelerating masses must be taken into consideration. In a given coordinate system the character of the inertia force for a shell element can be represented by a mass matrix. Instead of using the conventional lumped mass technique, we constructed the mass matrix by considering the correct mass distribution in the shell element. The technique of constructing the distributed or consistent mass matrix to associate it with the nodal rings is similar to the well-known Reyleigh-Ritz method for individual shell elements. The key step in this technique is to assume a displacement field for the shell element in terms of a certain coordinate system such as expressed by Eq. (4). In this report we will discuss two cases with different assumed displacement fields. In the first case, at a particular time t, the displacement field is assumed to be the same as that which was used in Part III for formulating the element stiffness matrix. For this purpose the time factor is introduced into Eq. (4), yielding

$$\begin{aligned} \left\{d(s,t)\right\} &= [X(s)] \begin{bmatrix} X\ (a) \\ \hline X\ (b) \end{bmatrix}^{-1} \left\{d(t)\right\} \\ &= [X(s)] \begin{bmatrix} X\ (a) \\ \hline X\ (b) \end{bmatrix}^{-1} [T] \left\{q(t)\right\} \end{aligned} \tag{12}$$

where the matrix $[X(s)]$ is defined in Part II. We call the mass matrix, which is obtained by using the above assumed displacement field, the "consistent mass matrix" for being consistent with the element stiffness matrix. In the second case, we assume the displacement field in the simplest possible polynomial form. In this case, the matrices $[X(s)]$ and $\left[\begin{array}{c} X(a) \\ \hline X(b) \end{array}\right]$ in Eq. (12) become

$$[X(s)] = \begin{bmatrix} 0 & 1 & 2s & 3s^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & s \\ 1 & s & s^2 & s^3 & 0 & 0 \end{bmatrix} \tag{13}$$

and

$$\left[\begin{array}{c} X(a) \\ \hline X(b) \end{array}\right] = \begin{bmatrix} 0 & 1 & 2a & 3a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & a \\ 1 & a & a^2 & a^3 & 0 & 0 \\ 0 & 1 & 2b & 3b^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b \\ 1 & b & b^2 & b^3 & 0 & 0 \end{bmatrix} \tag{13a}$$

We call the mass matrix, which is obtained by using the assumed polynomial displacement field, the "distributed mass matrix."

The general procedure of constructing the mass matrix is the same for the two cases. However, in determing the mass matrix a different meaning of $[X(s)]$ is assigned depending on whether the distributed or the consistent mass matrix is desired.

The general expression for the kinetic energy T(t) of a shell element can be written as

$$T(t) = \frac{1}{2}\int_s \left[m\ \rho_A^2\ \dot{\chi}^2\ (s,t) + m\ \dot{v}^2(s,t) + m\ \dot{w}^2\ (s,t)\right] 2\ \pi\ r(s) \quad ds \tag{14}$$

Where m is the mass per unit of shell surface area, ρ_A is the radius of gyration of the section of a shell element, and r(s) is the transverse radius of the shell element. For different basic shell elements see Figs. 1, 2 and 3.

Upon substituting the displacement variables involved in Eq. (12) into Eq. (14), we obtain

$$T(t) = \frac{1}{2} < \dot{q}(t) > \ [T]^T\ [B^{-1}]^T \int_s 2\pi\ [E(s)]\ r(s)ds\ [B^{-1}]\ [T]\ \left\{\dot{q}(t)\right\} \tag{15}$$

where

$$[E_{ij}(s)] = m\ \rho_A^2\ \left\{X_{1i}(s)\right\}\ < X_{1j}(s)> + m\ \left\{X_{2i}(s)\right\}\ < X_{2j}(s)>$$

$$+ m\ \left\{X_{3i}(s)\right\}\ < X_{3j}(s) > \tag{15a}$$

and

$i,\ j = 1,2,\ldots\ldots,6$

$< X_{1j}(s) >$, $< X_{2j}(s) >$, $< X_{3j}(s) >$ are the row matrices identical to the first, second and third row of the matrix [X(s)].

By comparing Eq. (15), with the usual one for kinetic energy,

$$T(t) = \frac{1}{2} < \dot{q}\ (t) > [m]\ \left\{\dot{q}(t)\right\} \tag{16}$$

We conclude that the element mass matrix [m] in the system coordinates is

$$[m] = [T]^T\ [B^{-1}]^T \int_s 2\pi\ [E(s)]\ r(s)\ ds\ [B^{-1}]\ [T] \tag{17}$$

The above equation can be used for formulating the element mass matrix in both cases, i.e., for the "consistent mass matrix" or for the "distributed mass matrix." For the "consistent mass matrix," the matrix [X(s)] from which the matrices [B] and [E(s)] can be obtained is defined in Articles II-1, II-2, and II-3, (see Eq. (10-a) and Eq. (15-a)) for the "distributed mass matrix" the matrix [X(s)] is defined by Eq. (13).

V. JOINT LOAD MATRIX FOR BASIC SHELL ELEMENTS

In the finite element analysis, only the forces and displacements of certain discrete (nodal) points of the structure are considered. At these points the compatibility and the equilibrium condition must be fulfilled. For this reason, any loading condition of a shell must be replaced by element joint loads which are considered to be a system equivalent to the actual loads. To accomplish this we assume that all of the actual load, which is distributed in the region between the centroid and upper end of the element, is concentrated or lumped at the upper end joint of the element, and all of the remaining load is lumped at the lower end joint of the element. Thus we can develop an expression for the "approximate joint load" $\left\{p(t)\right\}$ to represent the actual load $\left\{p(s,t)\right\}$, (see Fig. 6), i.e.,

$$\left\{p(t)\right\} = [T]^T \left\{ \begin{array}{c} \int_a^c p(s,t)\ 2\pi r(s)\ ds \\ \hdashline \int_c^b p(s,t)\ 2\pi r(s)\ ds \end{array} \right\} \tag{18}$$

where $[T]$ is the coordinate transformation matrix, defined by Eq. (8a) and a, b,c are the s values for the upper end, lower end and centroid of the shell element, respectively.

A solution using this "approximate joint load" matrix has proved to be quite satisfactory provided the size of the element is reasonably small (Ref. 3). However, if the size of the elements is made bigger, the discrepancy with the actual solution may become significant. In the latter cases, it may be desirable to use the more accurate equivalent joint load matrix, which we shall call "consistent joint load matrix." The "consistent joint load matrix" is constructed so that the work done by the actual load is equal to the work done by the "consistent joint load" due to a virtual displacement. Here we only write down the final results of the corresponding matrix equation. Further information on this matrix can be found in Ref. (15).

$$\left\{ p(t) \right\} = [T]^T \, [B^{-1}]^T \int_s [X(s)]^T \left\{ p(s,t) \right\} \, 2\pi r(s) \, ds \tag{19}$$

VI. THE EQUATION OF MOTION FOR THE DYNAMIC RESPONSE OF THE COMPLETE SHELL STRUCTURE

We have already set up shell element stiffness, shell element mass matrix, and shell element joint load matrix. We will now be able to find the relations between forces and displacements through structural stiffness matrix, structural mass matrix, and structural joint load matrix by means of continuity and equilibrium conditions at the nodal points. We consider that for a particular node, and in a particular direction, at any time the displacement of the structure equals the displacement of any element joint at that node. This continuity condition can be expressed by comparing structural displacement coordinate $\left\{r\right\}$ (Fig. 7) and element system displacement coordinate $\left\{q\right\}$ (Fig. 5) in the following equation:

$$\left\{q_i(t)\right\}_{(n)} = [\delta_{ij}]_{(n)} \left\{r_j(t)\right\} \tag{20}$$

where $\left\{q(t)\right\}_{(n)}$ is the element system displacement matrix for element (n), $\left\{r(t)\right\}$ is the structural displacement matrix, and $[\delta]_{(n)}$ is a transformation matrix for element (n). The latter matrix is defined as

$$[\delta_{ij}]_{(n)} = [0_{ik} \mid I_{i\ell} \mid 0_{im}] \tag{20a}$$

Where $[0_{ik}]$, $[0_{im}]$ are null submatrices, k = 1,2,...., 3 (n-1), m = 1,2,....., 3 (N-n), in which N is the total number of shell element, and $[I_{i\ell}]$ is a 6x6 identity submatrix.

In addition, the joint load of the structure at particular node equals the sum of the joint loads of the elements which meet at that node. Therefore, the structural joint load matrix $\left\{p(t)\right\}$ and the elements joint load matrices $\left\{P(t)\right\}$ have the following relation:

$$\left\{P_i(t)\right\} = \sum_{n=1}^{N} [\delta_{ji}]^T_{(n)} \left\{p_j\right\}_{(n)} \tag{21}$$

Where N again is the number of shell elements, $\left\{p(t)\right\}_{(n)}$ is the element joint load matrix for element n, and $\left\{P(t)\right\}$ is the structural joint load matrix.

Next, considering the n-th shell element only, we set-up the equation of motion for this shell element, that is:

$$\left\{p_i(t)\right\}_{(n)} = [k_{ij}]_{(n)} \left\{q_j(t)\right\}_{(n)} + [m_{ij}]_{(n)} \left\{\ddot{q}_j(t)\right\}_{(n)} \tag{22}$$

where the subscript (n) indicates that the matrix corresponds to the n-th element, and i, j = 1, 2,.....,6.

Substituting Eq. (20) and Eq. (22) into Eq. (21) gives the equation of motion of the system.

$$\left\{P_i(t)\right\} = \sum_{n=1}^{N} [\delta_{ij}]_{(n)} [k_{j\ell}]_{(n)} [\delta_{m\ell}]^T_{(n)} \left\{r_m(t)\right\} + \sum_{n=1}^{N} [\delta_{ij}]_{(n)} [m_{j\ell}]^T_{(n)} \left\{\ddot{r}_m(t)\right\} \tag{23}$$

where $i, m = 1, 2, \ldots\ldots, 3\ (n+1)$

$j, \ell = 1, 2, \ldots\ldots, 6$

or

$$\left\{P_i(t)\right\} = [K_{im}] \left\{r_m(t)\right\} + [M_{im}] \left\{\ddot{r}_m(t)\right\} \tag{24}$$

where [K] and [M] are respectively, the shell structure assemblage stiffness and mass matrices. These matrices are defined as follows

$$[K_{im}] = \sum_{n=1}^{N} [\delta_{ij}]_{(n)} [k_{j\ell}]_{(n)} [\delta_{m\ell}]^{T}_{(n)} \tag{24a}$$

$$[M_{im}] = \sum_{n=1}^{N} [\delta_{ij}]_{(n)} [m_{j\ell}]_{(n)} [\delta_{m\ell}]^{T}_{(n)} \tag{24b}$$

where $i, m = 1, 2, \ldots\ldots, 3\ (N+1).$, $j, \ell = 1, 2, \ldots\ldots 6$.

VII. ANALYSIS OF THE EQUATION OF MOTION

The equation of motion as expressed by Eq. (24), can be solved in many different ways. But if in addition to the dynamic responses, the free vibration characteristics of a shell are also to be determined, then it is advantageous to adopt the normal mode method of solution. This procedure is sometimes also called the "mode acceleration" method (see Ref. 14). The normal mode method is characterized by the fact that the differential equations of motion are uncoupled, where the displacements are expressed in terms of the normal modes. Therefore, in a system having n-degrees of freedom, we may deal with n independent differential equations rather than with a system of n simultaneous differential equations.

The structural assemblage mass matrix $[M]$, and the structural assemblage stiffness matrix $[K]$ in Eq. (24), are symmetrical and are positive definite. Therefore, Eq. 24 can be uncoupled (Ref. 12). This is to say that we always can find a matrix $[\Phi]$ such that

$$[\Phi]^T [M] [\Phi] = [I] \tag{25}$$

and

$$[\Phi]^T [K] [\Phi] = [\bar{\bar{\omega}}] \tag{26}$$

where $[\bar{\bar{\omega}}]$ is a diagonal matrix.

To find the matrix $[\phi]$, we first solve the eigenvectors and eigenvalues of the mass matrix $[M]$, such that

$$[\bar{\Phi}]^T [M] [\bar{\Phi}] = [\bar{\omega}] \tag{27}$$

where the columns of the matrix $[\bar{\Phi}]$ are the normalized eigenvectors of matrix $[M]$, and diagonal elements of the diagonal matrix $[\bar{\omega}]$ are the corresponding eigenvalues. Since $[M]$ is a positive definite, we have real, positive values for all $\bar{\omega}_i$ in $[\bar{\omega}]$. Thus, there always exists a real diagonal matrix $[\frac{1}{\sqrt{\bar{\omega}}}]$, and we can define a new matrix $[\bar{K}]$ such that

$$[\bar{K}] = [\frac{1}{\sqrt{\bar{\omega}}}] \; [\bar{\Phi}]^T [K] [\bar{\Phi}] [\frac{1}{\sqrt{\bar{\omega}}}] \tag{28}$$

Here again, $[\bar{K}]$ is a symmetrical, positive definite matrix. Therefore, it is possible to find matrices $[\bar{\bar{\Phi}}]$ and $[\bar{\bar{\omega}}]$ such that

$$[\bar{\bar{\Phi}}]^T [\bar{K}] [\bar{\bar{\Phi}}] = [\bar{\bar{\omega}}] \tag{29}$$

Where the columns of matrix $[\bar{\bar{\Phi}}]$ are normalized eigenvectors of $[\bar{K}]$, the diagonal elements $\bar{\bar{\omega}}_i$ of the diagonal matrix $[\bar{\bar{\omega}}]$ are the corresponding eigenvalues, and all $\bar{\bar{\omega}}_i$'s have non-repeated positive values, i.e.,

$$0 < \bar{\bar{\omega}}_1 < \bar{\bar{\omega}}_2 < \ldots\ldots < \bar{\bar{\omega}}_n$$

Furthermore, if we define a matrix $[\phi]$ such that

$$[\phi] = [\bar{\Phi}] [\frac{1}{\sqrt{\bar{\omega}}}] \; [\bar{\bar{\Phi}}] \tag{30}$$

then matrices $[\phi]$ and $[\bar{\bar{\omega}}]$ have the properties to satisfy Eqs. (25) and (26). This can be proved by substituting Eqs. (30), (28) and (29) into Eq. (25), and substituting Eqs. (30) and (27) into Eq. (26). To demonstrate,

$$[\phi]^T [K] [\phi] = [\bar{\bar{\phi}}]^T \left[\frac{1}{\sqrt{\bar{\omega}}} \right] [\bar{\phi}]^T [K] [\bar{\phi}] \left[\frac{1}{\sqrt{\bar{\omega}}} \right] [\bar{\bar{\phi}}]$$

$$= [\bar{\bar{\phi}}]^T [\bar{K}] [\bar{\bar{\phi}}]$$

$$= [\bar{\bar{\omega}}]$$

and

$$[\phi]^T [M] [\phi] = [\bar{\bar{\phi}}]^T \left[\frac{1}{\sqrt{\bar{\omega}}} \right] [\bar{\phi}]^T [M] [\bar{\phi}] \left[\frac{1}{\sqrt{\bar{\omega}}} \right] [\bar{\bar{\phi}}]$$

$$= [\bar{\bar{\phi}}]^T \left[\frac{1}{\sqrt{\bar{\omega}}} \right] [\bar{\omega}] \left[\frac{1}{\sqrt{\bar{\omega}}} \right] [\bar{\bar{\phi}}]$$

$$= [\bar{\bar{\phi}}]^T [\bar{\bar{\phi}}]$$

$$= [I]$$

In addition, we can assert that the columns of the matrix $[\phi]$ are the normal modes of the system, and the square root values of the diagonal elements $\bar{\bar{\omega}}_i$ of diagonal matrix $[\bar{\bar{\omega}}]$ are the corresponding frequencies.

The displacement vector $\{r(t)\}$ can be expressed in terms of the normal coordinates $\{\eta(t)\}$ as

$$\{r(t)\} = [\phi] \{\eta(t)\} \tag{31}$$

and the equation of motion can be uncoupled by substituting this relation into Eq. (24), and pre-multiplying it by matrix $[\phi]^T$. Thus,

$$[\phi]^T [K] [\phi] \left\{\eta(t)\right\} + [\phi]^T [M] [\phi] \left\{\ddot{\eta}(t)\right\} = [\phi]^T \left\{P(t)\right\}$$

or

$$\left\{\ddot{\eta}(t)\right\} + [\bar{\bar{\omega}}] \left\{\eta(t)\right\} = [\phi]^T \left\{P(t)\right\} \tag{32}$$

Note that the i^{th} row of Eq. (32) is the differential equation for the i^{th} mode which is independent of those for all other modes. Therefore, this equation may be integrated directly to yield the normal displacement $\eta_i(t)$. This may be repeated independently for all other modes to solve for the n normal displacements $\left\{\eta_i(t)\right\}$, i = 1,2,.....n. The total displacement of the structure $\left\{r(t)\right\}$ is obtained by inserting the solved $\left\{\eta(t)\right\}$ values into Eq. (31).

VIII. INTERNAL STRESS-RESULTANTS RESPONSE

Theoretically, the internal stress resultants $\left\{Q(t)\right\}$ of the shell element n may be obtained by using element stiffness $[k]_{(n)}$ directly. That is

$$\begin{aligned}\left\{Q(t)\right\}_{(n)} &= [k]_{(n)} \left\{q(t)\right\}_{(n)} \\ &= [k]_{(n)} [\delta]^{T}_{(n)} \left\{r(t)\right\} \qquad (33) \\ &= [k]_{(n)} [\delta]^{T}_{(n)} [\phi] \left\{\eta(t)\right\}\end{aligned}$$

Actually, the above expression for $\left\{Q(t)\right\}$ leads to highly inaccurate results.

In using Eq. (33) to calculate the internal stress-resultant response $\left\{Q(t)\right\}$ an acceptable computational error in the displacement response $\left\{r(t)\right\}$ is greatly amplified by stiffness matrix $[k]$. Following the suggested procedure given in Ref. 16, we adopt an alternative method of analysis here so that the error due to the sensitivity of the stress resultants $\left\{Q(t)\right\}$ to computational errors can be kept small, and the degree of accuracy achieved will be of the same order as that of the displacement response $\left\{r(t)\right\}$.

In this method the internal stress resultants are computed in two parts. First, we apply external load $\left\{P(t)\right\}$ statically, and compute the internal forces under the assumption that all points of the system have zero acceleration. This can be done by applying to the system the total exciting force $\left\{P(t)\right\}$ at a time t, and computing the total static internal resultants. We designate these internal resultants by $\left\{Q(t)\right\}_{I}$. Then, we add to this the internal resultants associated with the acceleration of the system. To obtain the internal forces $\left\{Q(t)\right\}_{II}$ corresponding to the acceleration of the system, we refer to the uncoupled differential equation of motion for the i^{th} mode, i.e., we consider, for example, the i^{th} row of Eq. (32)

$$\ddot{\eta}_i(t) + \bar{\bar{\omega}}_i \; \eta_i(t) = P_i^*(t) \tag{34}$$

This equation can be written in the form

$$\eta_i(t) = \frac{P_i^*(t)}{\bar{\bar{\omega}}_i} - \frac{\ddot{\eta}_i(t)}{\bar{\bar{\omega}}_i} \tag{35}$$

or

$$-\frac{\ddot{\eta}_i(t)}{\bar{\bar{\omega}}_i} = \eta_i(t) - \frac{P_i^*(t)}{\bar{\bar{\omega}}_i} \tag{36}$$

The term $\dfrac{P_i^*(t)}{\bar{\bar{\omega}}_i}$ in Eq. (35) represents the response due to loads $P_i^*(t)$ applied to the system statically, namely for $\ddot{\eta}_i = 0$. The term $-\dfrac{\ddot{\eta}_i}{\bar{\bar{\omega}}_i}$ in Eq. (35) represents the response due to the acceleration $\ddot{\eta}_i(t)$ of the system when it is vibrating in its i^{th} mode.

To obtain the internal resultants $\left\{Q(t)\right\}_{II}$ corresponding to the acceleration of the system, we use Eq. (36), where all $-\dfrac{P_i^*(t)}{\bar{\bar{\omega}}_i}$ and $\eta_i(t)$ have known values. For each normal mode there corresponds an external joint load system, which when applied to the structure, will cause it to deform in its i^{th} mode with an amplitude of unity ($\eta_i(t) = 1$). For the i^{th} mode this joint loading is given by the following joint inertial loads:

$$\text{joint load matrix} = \bar{\bar{\omega}}_i \; [M] \; \left\{ \phi \right\}_i \tag{37}$$

Where $\left\{ \phi \right\}_i$ is the i^{th} normal mode of the system, i.e., the i^{th} column of matrix $[\phi]$. Using this inertia load we can compute the corresponding internal resultants $\left\{ Q(t) \right\}_i$. Then these resultants are amplified by the value of the i^{th} normal mode acceleration response $-\dfrac{\ddot{\eta}_i(t)}{\bar{\bar{\omega}}_i}$, and we obtain the internal resultants $\left\{ Q(t) \right\}_{II}$ corresponding to the acceleration of the system. That is

$$\left\{ Q(t) \right\}_{II} = \sum_i - \frac{\ddot{\eta}_i(t)}{\bar{\bar{\omega}}_i} \left\{ Q(t) \right\}_i \tag{38}$$

where $\left\{ Q(t) \right\}_i$ are the internal resultants corresponding to the joint inertial load of the i^{th} normal mode. The total internal resultants are given by

$$\left\{ Q(t) \right\} = \left\{ Q(t) \right\}_I + \left\{ Q(t) \right\}_{II} \tag{39}$$

Where, as defined earlier, $\left\{ Q(t) \right\}_I$ represent the internal resultants due to the total external force exciting the system, when this force is applied statically. In this approach the internal resultants $\left\{ Q(t) \right\}_{II}$ are obtained from the inertial joint loads associated with each joint. This procedure tends to reduce the degree of error that may result from the discrepancies in the inertial loads computed from Eq. (37) due to the fact that the modes $\left\{ \phi \right\}_i$ are approximate.

In Eq. (39) $\{Q(t)\}_I$ represent the internal resultants due to all externally applied forces exciting the system, when such forces are applied statically. Therefore, irrespective of the number of modes considered in the analysis, the resultants $\{Q(t)\}_I$ are accurately determined. If Eq. (33) were used instead, the static effect of the applied force $\{P(t)\}$ depends on the number of modes considered in the analysis. Hence, if a small number of modes is involved in the analysis, the static effect of the applied forces $\{P(t)\}$ is not completely accounted for if Eq. (33) is used.

IX. EXAMPLES AND CONCLUSIONS

As the first example, consider an elastic circular plate clamped along the outer edge subjected to a ring load as shown in Fig. 8 for which its dynamic response is to be determined. The ring load P is applied as a step function in time. This particular problem has been solved using the "consistent mass matrix" in the equation of motion. To determine the dynamic response of this plate by the developed method, only two elements need to be used, since the element mass matrix [m] is consistent with the exact element stiffness matrix. Alternatively, an arbitrary number of elements may be used, and 20 elements were selected to obtain a solution for comparative purposes. The results of the two solutions are plotted in Figs. 11b and 11c. Differences between the two solutions are negligible. The solution based on the use of 20 elements actually is a little less accurate due to the unavoidable accumulation of numerical errors. In Fig. 11a the first three normalized modes and the corresponding frequencies are given. Considering the rapid raise of the frequencies and the nature of the corresponding mode shapes, the response due to the effect of the second and third modes is minimal.

The second example is of the dynamic response of the shallow spherical shell shown in Fig. 9. The data are from the Klein and Sylvester example (Ref. 5). The 26.67° sphere was analyzed as an assemblage of a 0.67° spherical cap and 14 cones. "Distributed mass matrix" and "Approximate joint loads" were used to obtain the solution for this problem. The dynamic and results are shown in Fig. 12b, 12c and 12d. These results are seen to be in excellent agreement with the Klein and Sylvester solution. Also presented is a plot of the first three mode shapes and their corresponding frequencies, Fig. 12a. In this problem, due to high frequencies and slow convergence, the effect of second and third modes is as important as that of the first mode on the response of the system. Results were also obtained by using only the first three modes instead of the 16 modes as above. This led to poor results. The discrepancy between the two solutions is significant, and the solution with three modes is unacceptable. From the experience gained in solving different shell dynamic problems by using the developed computer program it is concluded that for deep shell structures at least 20 modes have to be used to obtain satisfactory results. For very shallow shells, on the other hand, 3 modes may give resonable results. The dynamic response of the sphere shown in Fig. 10 was solved by using 50 elements. A plot of the normal displacement at a point where the ring load P is applied is presented in Fig. 13.

The dynamic response of linear elastic shells of revolution of arbitrary meridian shape and thickness variation can be determined using the finite element approach. The accuracy appears to be excellent, and once a program is developed a solution is achieved very rapidly. In the cases when the shell is actually a combination of the basic shell elements used in this report, and the dynamic loads are localized then the solution based on a mass matrix which is consistent with the exact stiffness matrix may prove particularly advantageous.

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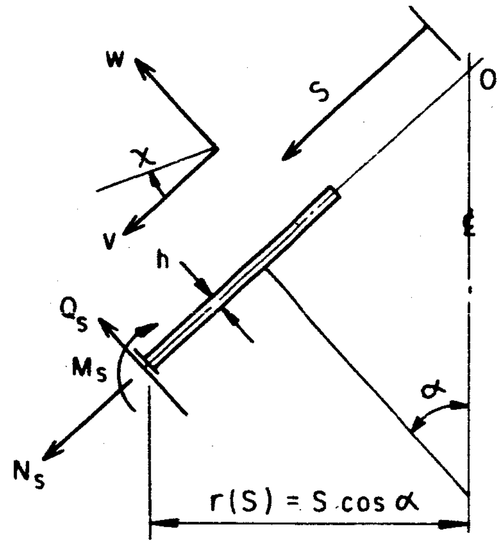


h = THICKNESS

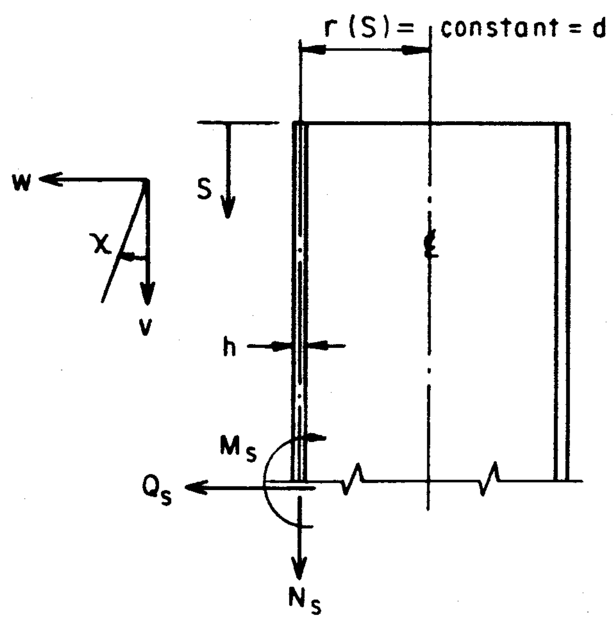
ν = POISSON RATIO

E = ELASTIC MODULUS

S = DISTANCE FROM APEXO

$$y = 2\sqrt[4]{3(1-\nu^2)}\,\sqrt{\frac{2\tan\alpha}{h}}\,\sqrt{S}$$

FIG. 1 CONICAL ELEMENT FORCE AND DISPLACEMENT VARIABLES

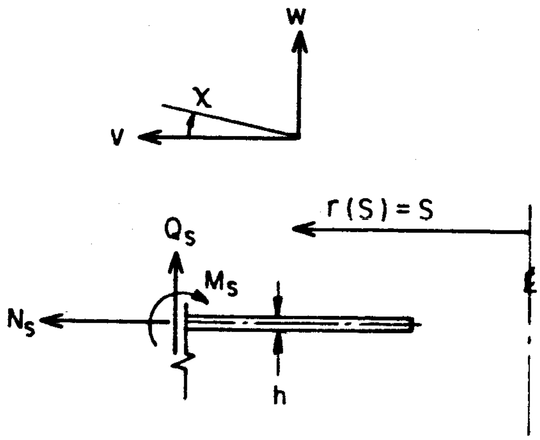


$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$K^4 = 3(1-\nu^2)\,\frac{d^2}{h^2}$$

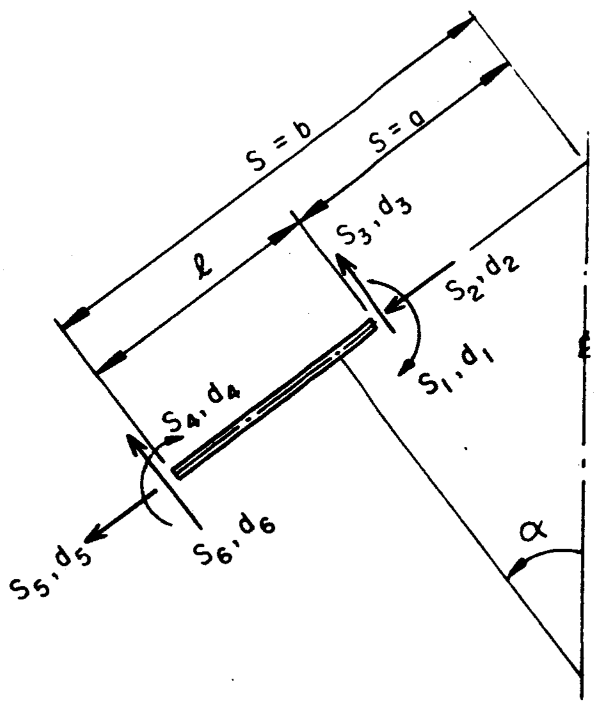
$$\xi = \frac{S}{d}$$

FIG. 2 CYLINDRICAL ELEMENT FORCE AND DISPLACEMENT VARIABLES



$$D = \frac{Eh^3}{12(1-\nu^2)}$$

FIG. 3 PLATE ELEMENT FORCE AND DISPLACEMENT VARIABLES



FOR CYLINDRICAL ELEMENT

$\alpha = 90°$

$a = 0 \quad b = \ell$ (See Fig. 2)

FOR PLATE ELEMENT

$\alpha = 0°$

FIG. 4 BASIC SHELL ELEMENT FORCE AND DISPLACEMENT LOCAL COORDINATE

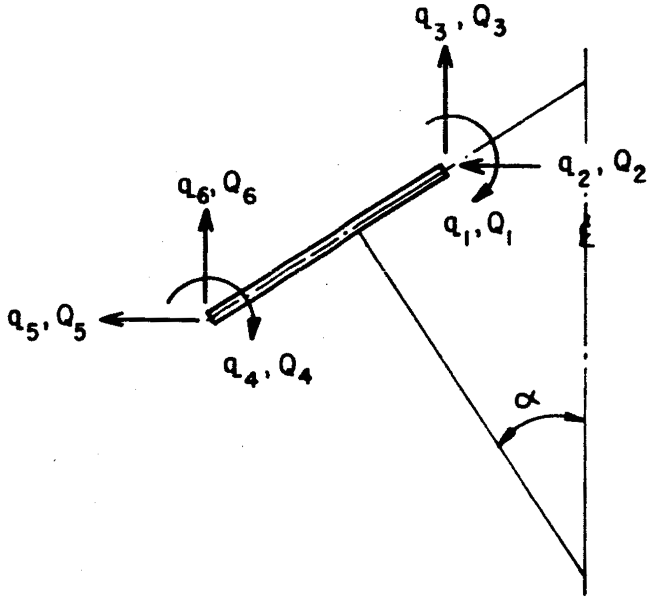
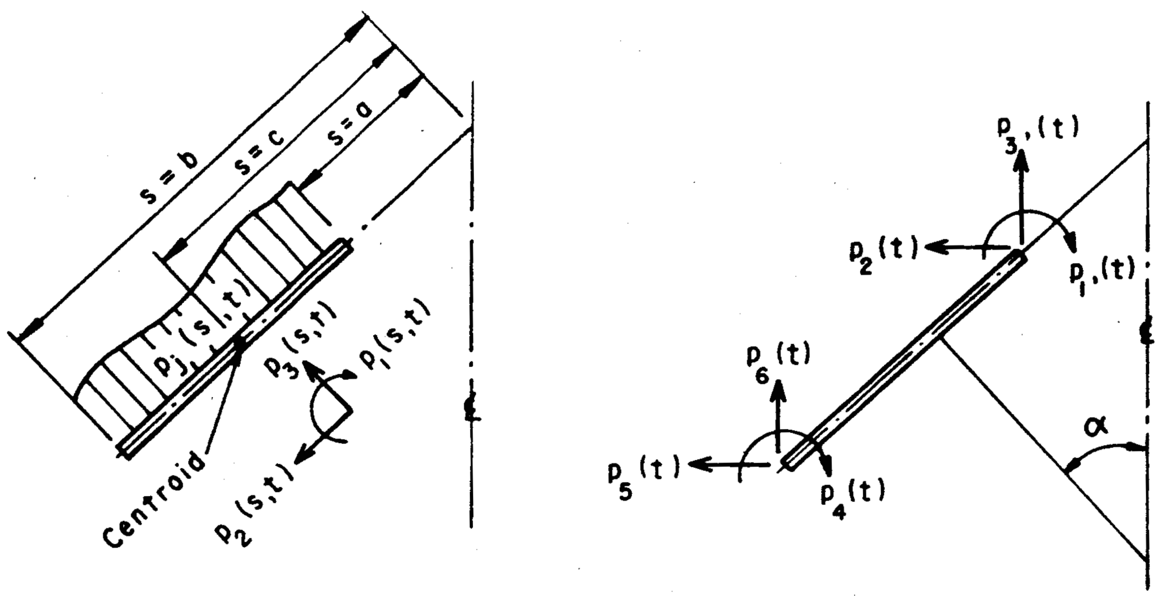


FIG. 5 BASIC SHELL ELEMENT FORCE AND DISPLACEMENT GLOBAL COORDINATE



ACTUAL LOAD $\{p_j(s,t)\}$

JOINT LOAD $\{p_i(t)\}$

FIG. 6 SHELL ELEMENT EXTERNAL FORCES

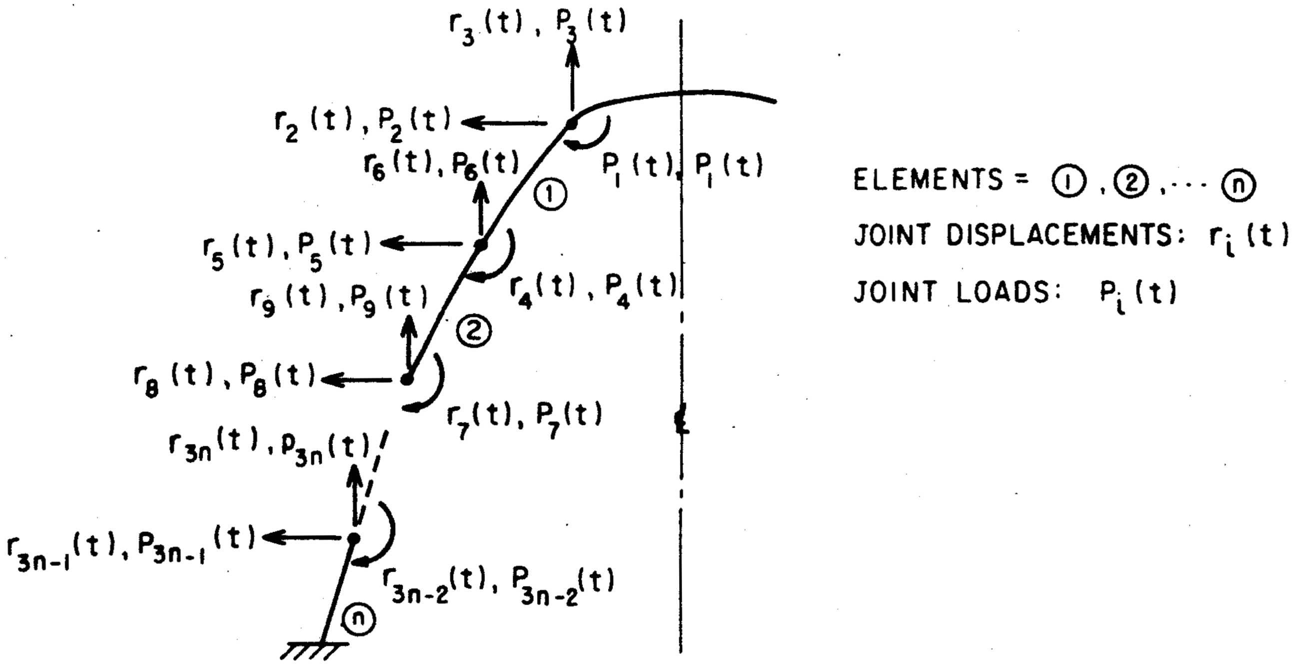


FIG. 7 STRUCTURAL JOINT LOADS AND JOINT DISPLACEMENTS

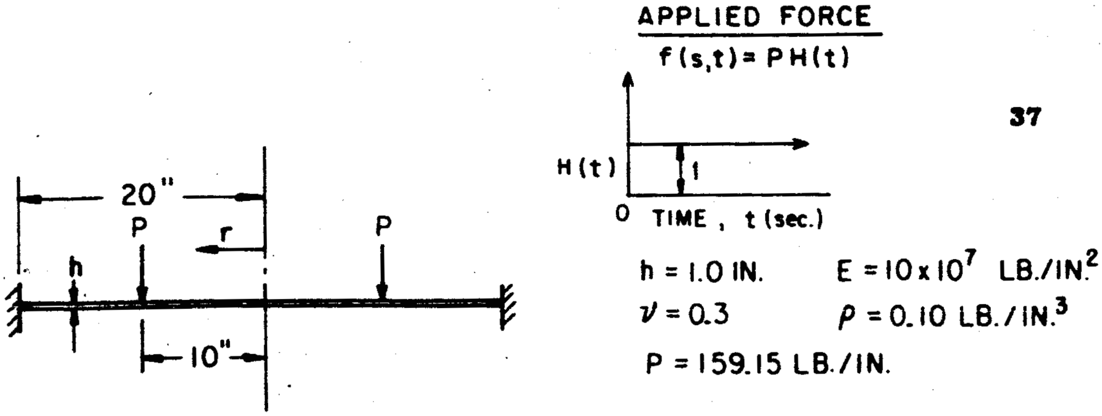


FIG. 8 A CIRCULAR PLATE UNDER DYNAMIC LOADING

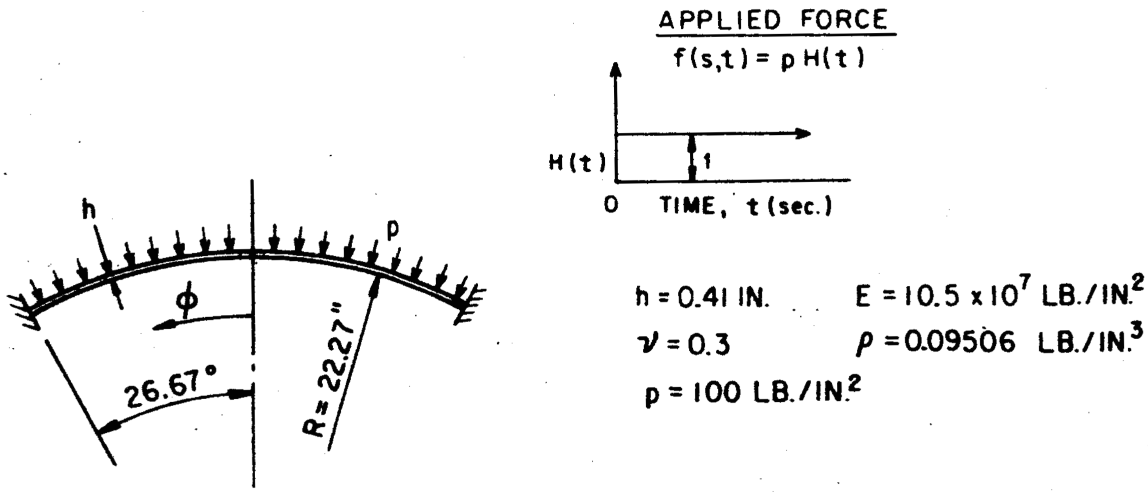


FIG. 9 A SPHERICAL CAP UNDER DYNAMIC LOADING

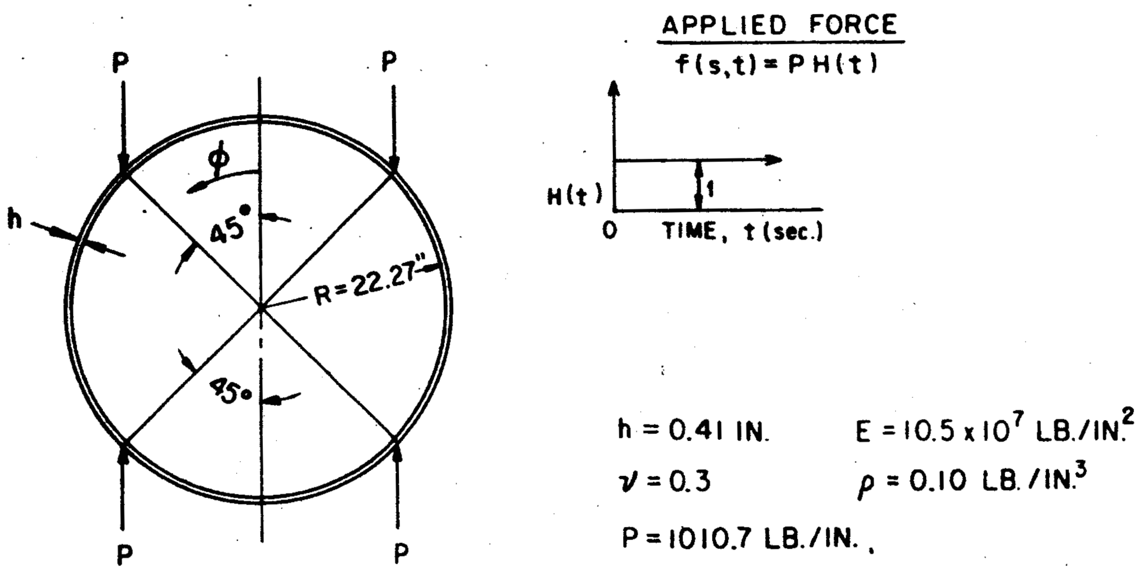


FIG. 10 A SPHERE UNDER DYNAMIC LOADING

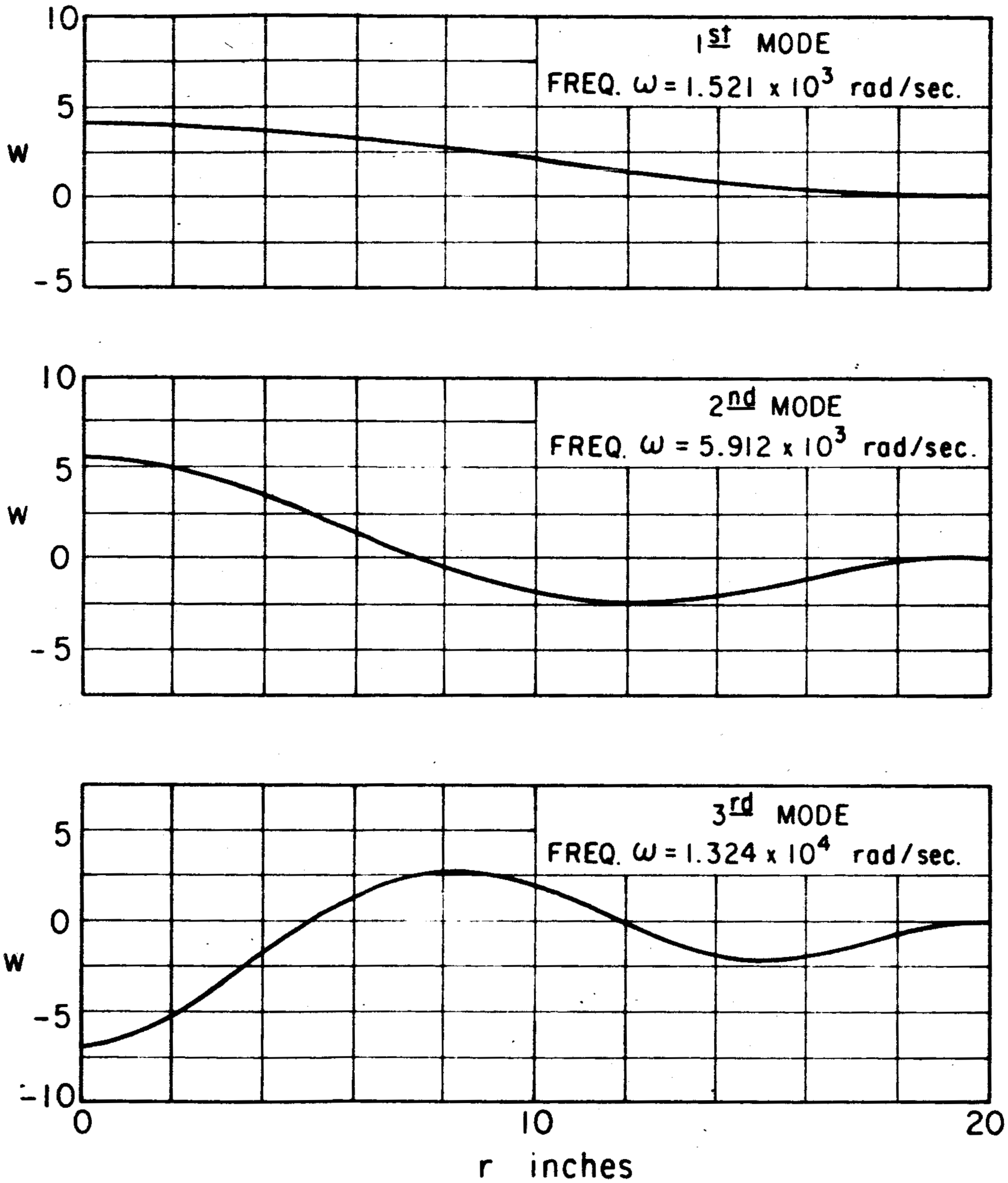


FIG. 11a NORMALIZED MODE SHAPES OF VERTICAL DISPLACEMENT w FOR THE CIRCULAR PLATE

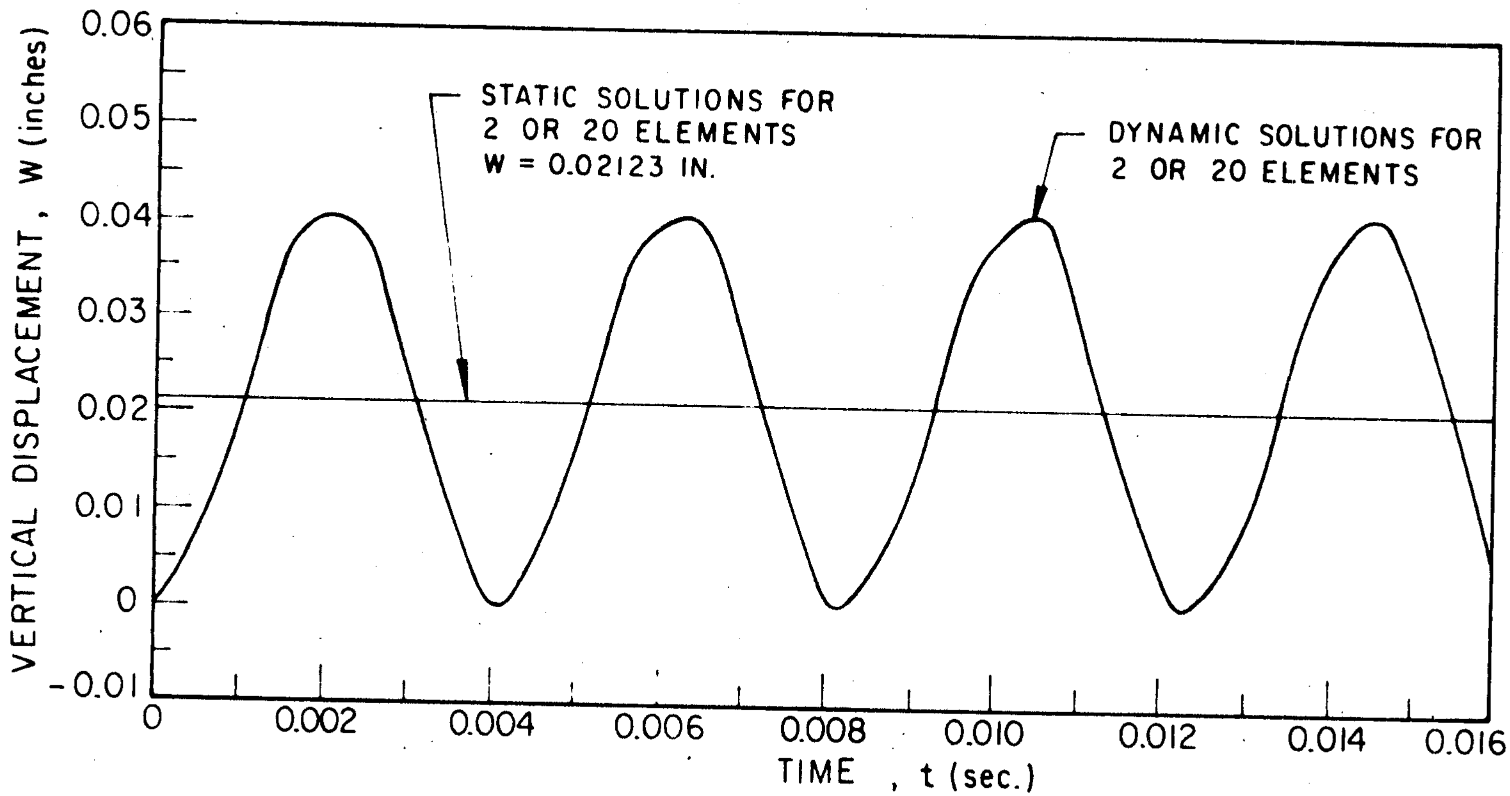


FIG. 11b VERTICAL DISPLACEMENT RESPONSE AT r = 10 IN. FOR THE CIRCULAR PLATE

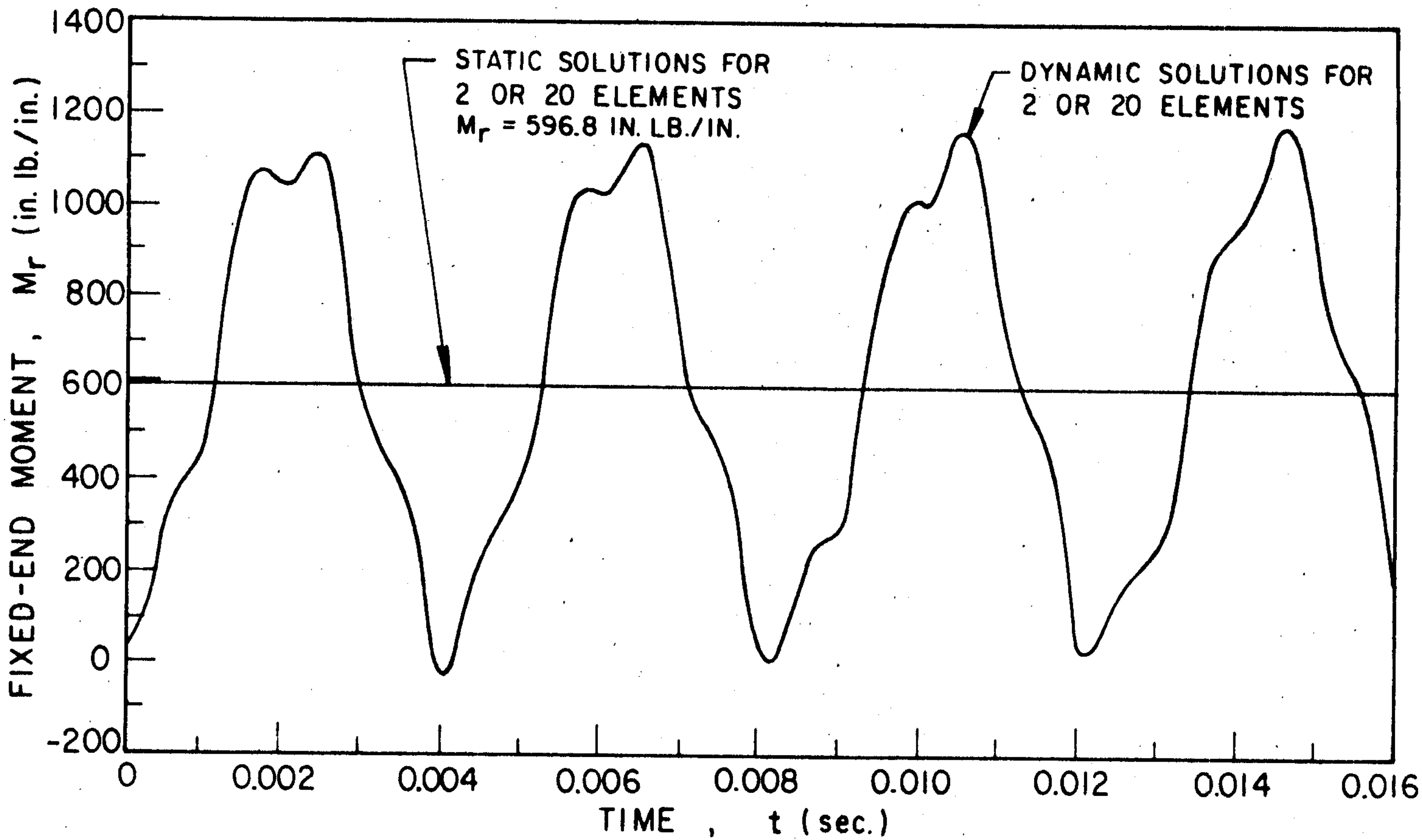


FIG. 11c FIXED-END MOMENT RESPONSE AT r = 20 IN. FOR THE CIRCULAR PLATE

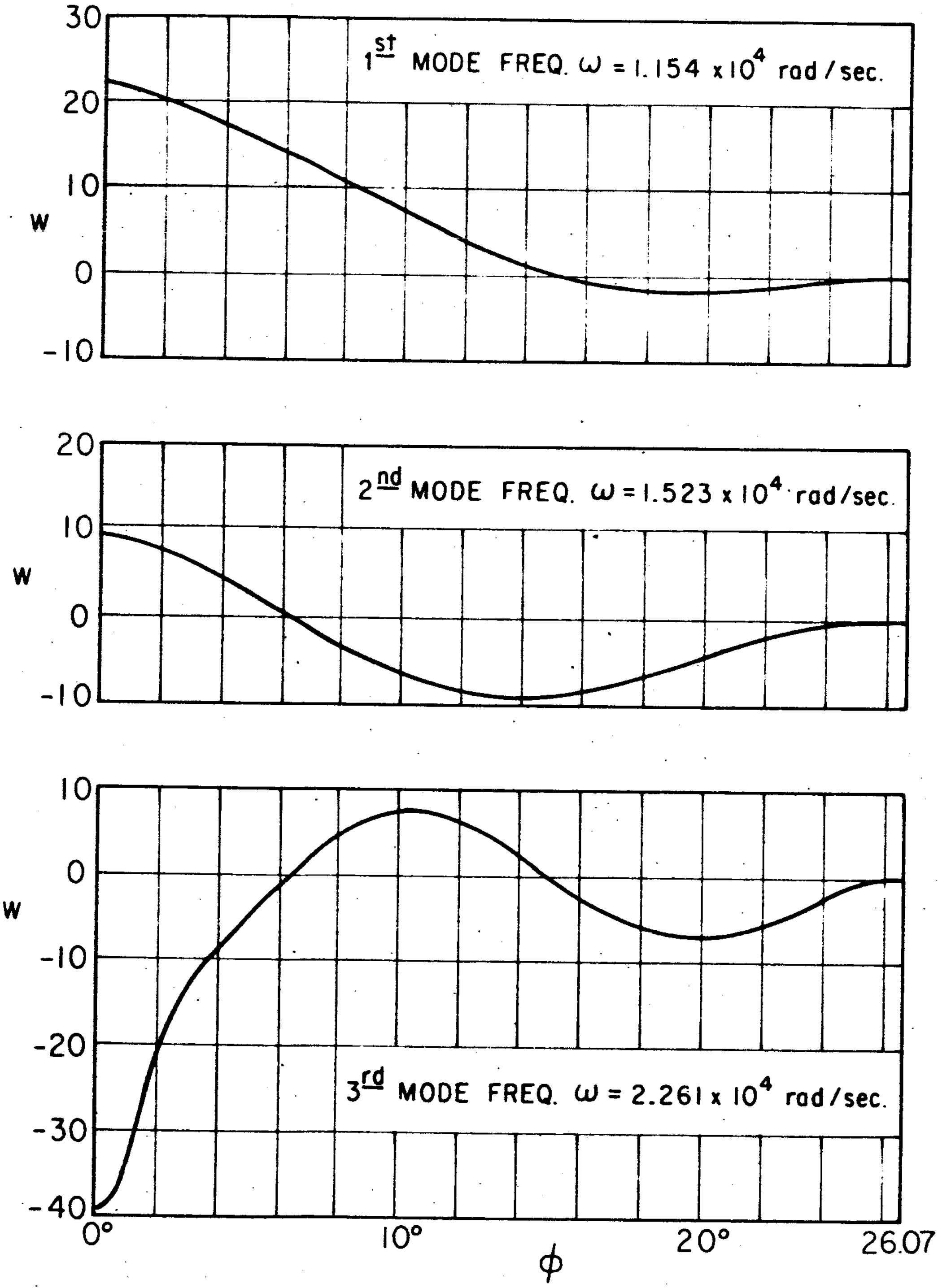


FIG. 12 a NORMALIZED MODE SHAPES OF NORMAL DISPLACEMENT w FOR THE SPHERICAL CAP.

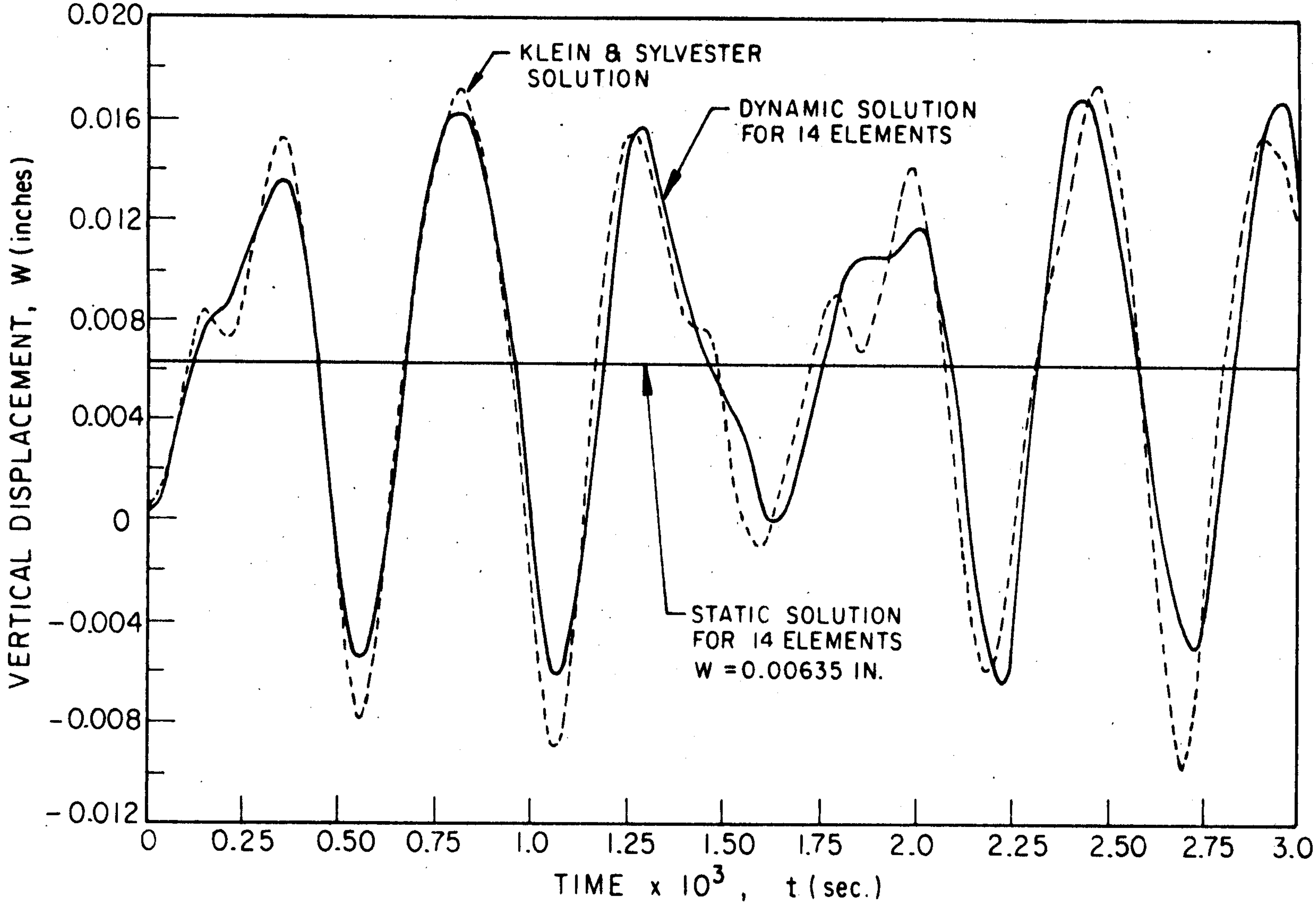


FIG. 12b VERTICAL DISPLACEMENT RESPONSE AT $\phi = 0°$ FOR THE SPHERICAL CAP

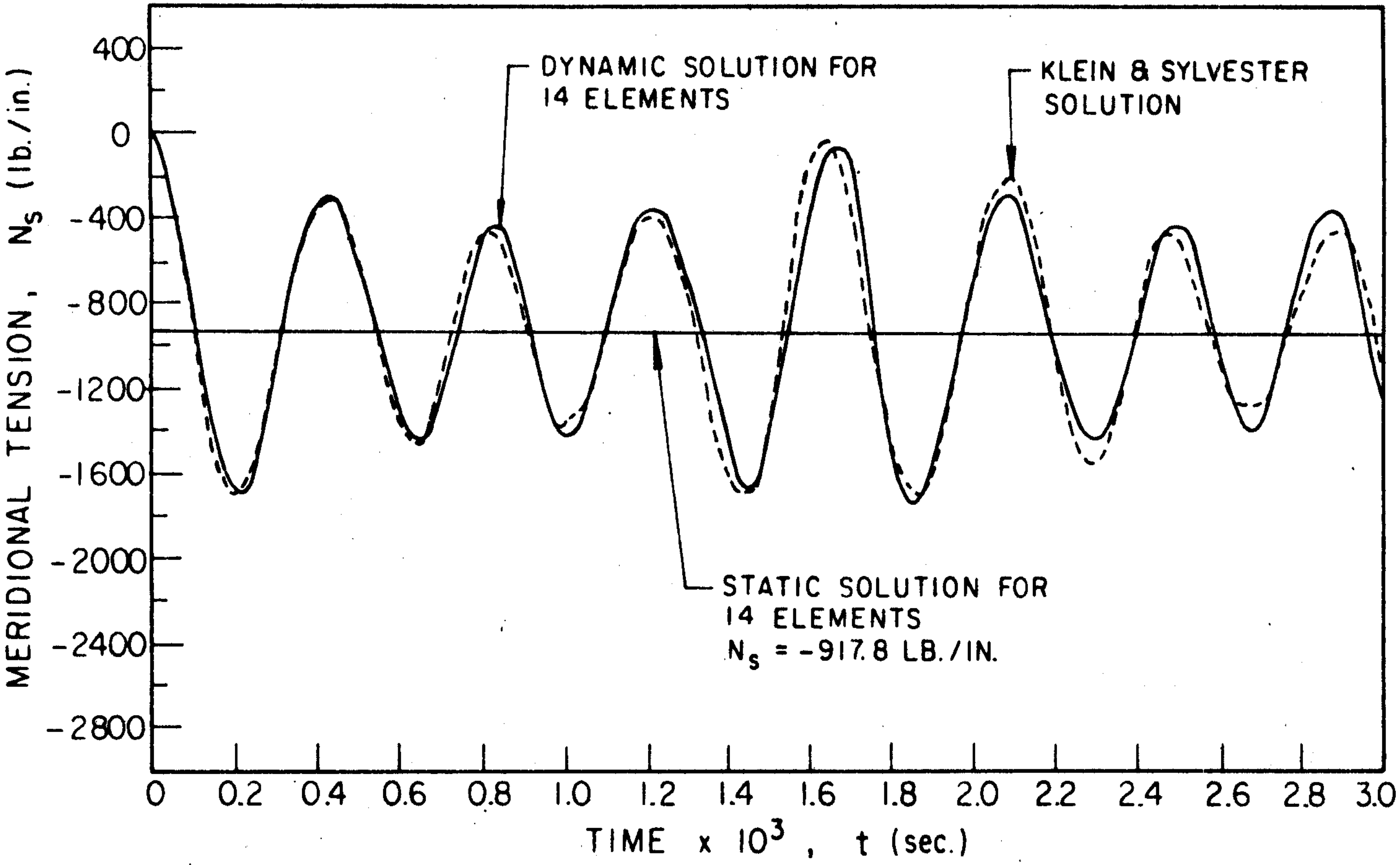


FIG. 12c MERIDIONAL TENSION RESPONSE AT $\phi = 26.67°$ FOR THE SPHERICAL CAP

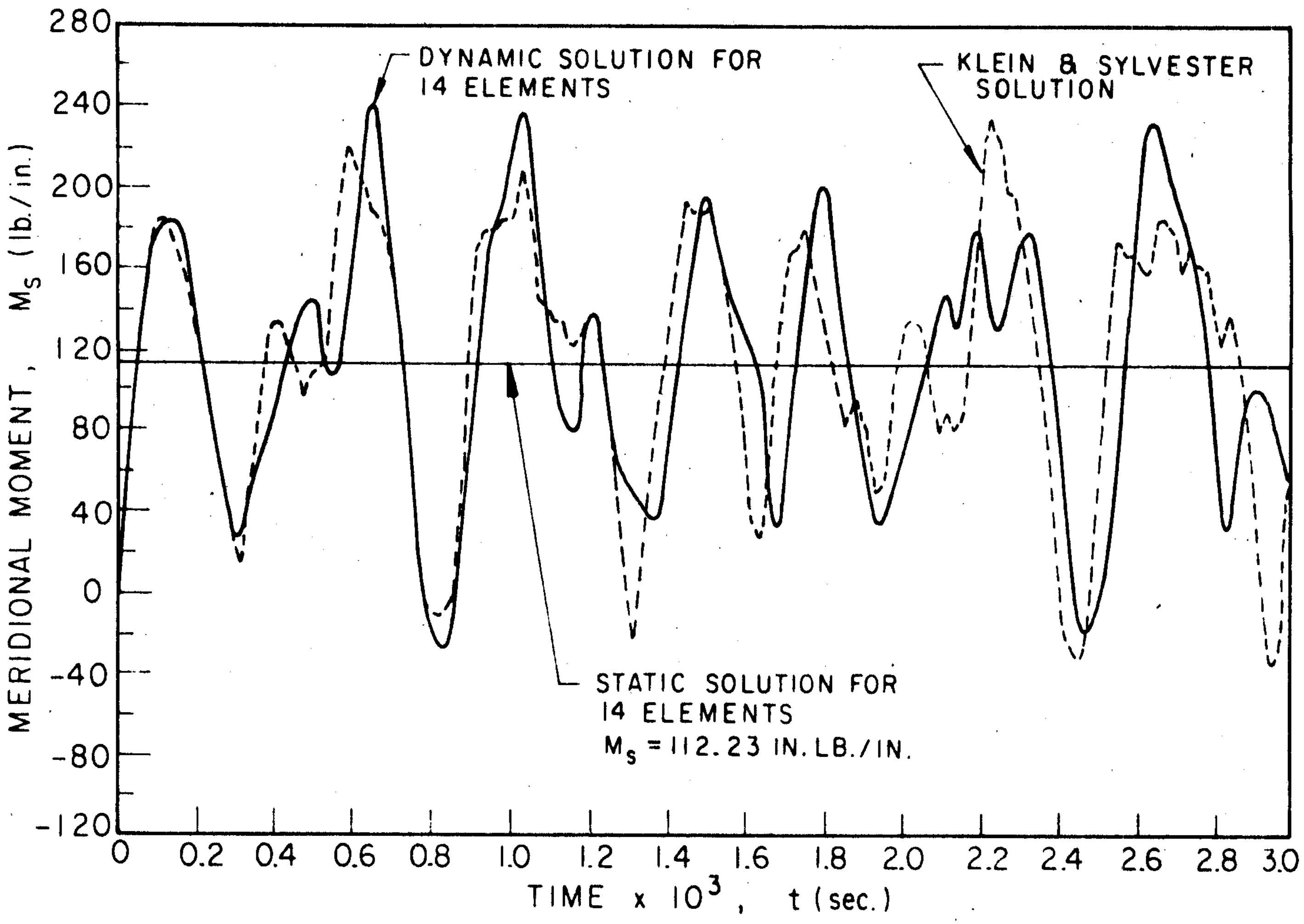


FIG. 12d MERIDIONAL MOMENT RESPONSE AT ϕ = 26.67°
FOR THE SPHERICAL CAP

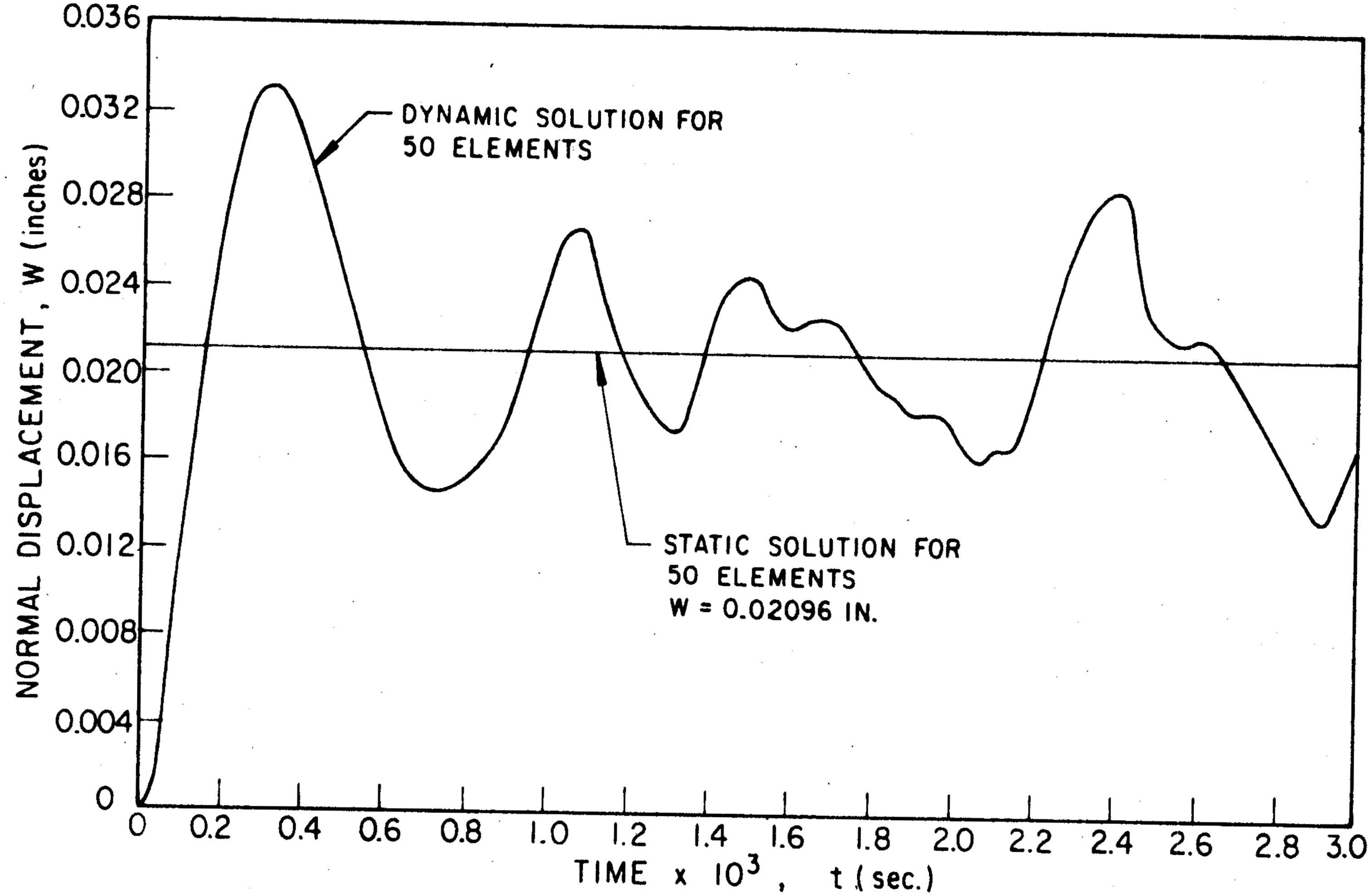


FIG. 13 NORMAL DISPLACEMENT RESPONSE AT $\phi = 45°$ FOR THE SPHERE